

Some Evaluation of Quadratic Euler Sums

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Abstract In this paper, we obtain some formulas for double nonlinear Euler sums involving harmonic numbers and alternating harmonic numbers. By using these formulas, we give new closed form sums of several quadratic Euler series through Riemann zeta values, polylogarithm functions and linear sums. Furthermore, some relationships between Euler sums and integrals of polylogarithm functions are established.

Keywords Polylogarithm function; Euler sum; Riemann zeta function.

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1 Introduction

Throughout this article we will use the following definitions and notations. In this paper, the generalized harmonic numbers and alternating harmonic numbers is defined by

$$H_n = \sum_{j=1}^n \frac{1}{j}, \quad \zeta_n(k) = \sum_{j=1}^n \frac{1}{j^k}, \quad L_n(k) = \sum_{j=1}^n \frac{(-1)^{j-1}}{j^k}, \quad 1 \leq k \in \mathbb{Z}. \quad (1.1)$$

For a pair (p, q) of positive integers with $q \geq 2$, the classical double linear Euler sum is defined by

$$S_{p,q} = \sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{k=1}^n \frac{1}{k^p}, \quad (1.2)$$

The number $w = p + q$ is defined as the weight of $S_{p,q}$. The evaluation of $S_{p,q}$ in terms of values of Riemann zeta function at positive integers is known when $p = 1$, $p = q$, $(p, q) = (2, 4), (4, 2)$ or $p + q$ is odd. In 1742, Goldbach proposed to Euler the problem of expressing the $S_{p,q}$ in terms of values at positive integers of the Riemann zeta function $\zeta(s)$. The Riemann zeta function and alternating Riemann zeta function are defined respectively by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1,$$

and

$$\bar{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \Re(s) \geq 1.$$

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Euler proved that the double linear sums are reducible to zeta values whenever $a + b$ is less than 7 or when $a + b$ is odd and less than 13. He conjectured that the double linear sums would be reducible to zeta values when $p + q$ is odd, and even gave what he hoped to obtain the general formula. In [3], D. Borwein, J.M. Borwein and R. Girgensohn proved conjecture and formula, and in [2], D.H. Bailey, J.M. Borwein and R. Girgensohn conjectured that the double linear sums when $p + q > 7$, $p + q$ is even, are not reducible. In [12], Philippe Flajolet and Bruno Salvy gave a general formula for odd weight $p + q$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q} &= \zeta(m) \left(\frac{1}{2} - \frac{(-1)^p}{2} \binom{m-1}{p} - \frac{(-1)^p}{2} \binom{m-1}{q} \right) \\ &\quad + (-1)^p \sum_{k=1}^{\lfloor p/2 \rfloor} \binom{m-2k-1}{q-1} \zeta(2k) \zeta(m-2k) + \frac{1-(-1)^p}{2} \zeta(p) \zeta(q) \\ &\quad + (-1)^p \sum_{k=1}^{\lfloor q/2 \rfloor} \binom{m-2k-1}{p-1} \zeta(2k) \zeta(m-2k), \end{aligned}$$

where $\zeta(1)$ should be interpreted as 0 wherever it occurs.

Let $\pi = (\pi_1, \dots, \pi_k)$ be a partition of integer p and $p = \pi_1 + \dots + \pi_k$ with $\pi_1 \leq \pi_2 \leq \dots \leq \pi_k$. The classical double nonlinear Euler sum of index π, q is defined as follows (see [12])

$$S_{\pi, q} = \sum_{n=1}^{\infty} \frac{\zeta_n(\pi_1) \zeta_n(\pi_2) \cdots \zeta_n(\pi_k)}{n^q}, \quad (1.3)$$

where the quantity $\pi_1 + \dots + \pi_k + q$ is called the weight, the quantity k called the degree. As usual, repeated summands in partitions are indicated by powers, so that for instance

$$S_{1^2 2^3 4, q} = S_{112224, q} = \sum_{n=1}^{\infty} \frac{H_n^2 \zeta_n^3(2) \zeta_n(4)}{n^q}.$$

The general Euler sums are defined by the series (see [19-21])

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod_{j=1}^{m_2} L_n^{l_j}(h_j)}{n^p}, \quad \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod_{j=1}^{m_2} L_n^{l_j}(h_j) (-1)^{n-1}}{n^p}, \quad (1.4)$$

where $p(p > 1), m_1, m_2, q_i, k_i, h_j, l_j$ are positive integer. If $\sum_{i=1}^{m_1} (k_i q_i) + \sum_{j=1}^{m_2} (h_j l_j) + p = C$ (C is a positive integer), then we call the identity C th-order Euler sums.

In [13], Philippe Flajolet and Bruno Salvy gave explicit reductions to zeta values and logarithm for all linear sums

$$\sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q}, \quad \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q}, \quad \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q} (-1)^{n-1}, \quad \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q} (-1)^{n-1}$$

when $p + q$ is an odd weight. The evaluation of linear sums in terms of values of the Riemann zeta function and polylogarithm function at positive integers is known when $(p, q) = (1, 3), (2, 2)$, or $p + q$ is odd. For example

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} (-1)^{n-1} = -2Li_4\left(\frac{1}{2}\right) + \frac{11}{4}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - \frac{7}{4}\zeta(3)\ln 2,$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L_n(1)}{n^3} (-1)^{n-1} &= \frac{3}{2} \zeta(4) + \frac{1}{2} \zeta(2) \ln^2 2 - \frac{1}{12} \ln^4 2 - 2Li_4\left(\frac{1}{2}\right), \\
\sum_{n=1}^{\infty} \frac{L_n(2)}{n^2} &= \frac{85}{16} \zeta(4) - 4Li_4\left(\frac{1}{2}\right) + \zeta(2) \ln^2 2 - \frac{1}{6} \ln^4 2 - \frac{7}{2} \zeta(3) \ln 2, \\
\sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^2} (-1)^{n-1} &= -\frac{51}{16} \zeta(4) + 4Li_4\left(\frac{1}{2}\right) + \frac{7}{2} \ln 2 \zeta(3) - \zeta(2) \ln^2 2 + \frac{\ln^4 2}{6}.
\end{aligned}$$

In many other cases we are not able to obtain a formula for the Euler sum constant explicitly in terms of values of the Riemann zeta, logarithm and polylogarithm functions, but we are able to obtain relations involving two of more Euler sum constants of the same degree. In [21], we proved that the quadratic double sums

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}}, \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}}$$

are reducible to polynomials zeta values and to linear sums, m is a positive integer. The relationship between the values of the Riemann zeta function and Euler sums has been studied by many authors, for example see [2-4,6-21].

The main purpose of this paper is to evaluate some quadratic Euler sums which involving harmonic numbers and alternating harmonic numbers, either linearly or nonlinearly. In this paper, we will prove that all quadratic double sums

$$\sum_{n=1}^{\infty} \frac{\zeta_n(p) \zeta_n(p+2m+1)}{n} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{L_n(p) L_n(p+2m+1)}{n} (-1)^{n-1}, \quad 2 \leq p \in \mathbb{Z}, \quad 0 \leq m \in \mathbb{Z}$$

are reducible to linear sums and to polynomials zeta values. In the same way, we also obtain that, for $2 \leq p \in \mathbb{Z}$, $0 \leq m \in \mathbb{Z}$, the following expression

$$\sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(p+2m+1)}{n^p} + \frac{H_n \zeta_n(p)}{n^{p+2m+1}} \right\}$$

and

$$\sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(p+2m+2)}{n^p} - \frac{H_n \zeta_n(p)}{n^{p+2m+2}} \right\}$$

are reducible to linear sums.

2 Main Theorems and Proof

In this section, we will establish some explicit relationships which involve Euler sums and integrals of polylogarithm functions. The polylogarithm function defined as follows

$$Li_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \quad \Re(p) > 1, \quad |x| \leq 1.$$

when x takes 1 and -1 , then the function $Li_p(x)$ are reducible to Riemann zeta function and alternating Riemann zeta function, respectively.

Lemma 2.1 *Let $s, t \geq 1$ be integers. Then the product $Li_s(x) Li_t(x)$ are reducible to Euler sums function:*

$$Li_s(x) Li_t(x) = \sum_{j=1}^s A_j^{(s,t)} \sum_{n=1}^{\infty} \frac{\zeta_n(j)}{n^{s+t-j}} x^n + \sum_{j=1}^t B_j^{(s,t)} \sum_{n=1}^{\infty} \frac{\zeta_n(j)}{n^{s+t-j}} x^n - \left(\sum_{j=1}^s A_j^{(s,t)} + \sum_{j=1}^t B_j^{(s,t)} \right) Li_{s+t}(x). \quad (2.1)$$

where $A_j^{(s,t)} = \binom{s+t-j-1}{s-j}$, $B_j^{(s,t)} = \binom{s+t-j-1}{t-j}$.

Proof. The left hand of equation (2.1) equals

$$Li_s(x) Li_t(x) = \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{x^{n+1}}{k^s (n-k+1)^t}. \quad (2.2)$$

By the formula

$$\frac{1}{x^s(1-x)^t} = \sum_{j=1}^s \frac{A_j^{(s,t)}}{x^j} + \sum_{j=1}^t \frac{B_j^{(s,t)}}{(1-x)^j}, \quad s, t \geq 0, s+t \geq 1. \quad (2.3)$$

we can obtain (2.1).

Theorem 2.2 *Let $p, q \geq 1$ be integers and $x \in [-1, 1)$. Then the following identity holds*

$$\int_0^x \frac{Li_p(t) Li_q(t)}{t} dt = \sum_{i=1}^{q-1} (-1)^{i-1} Li_{p+i}(x) Li_{q+1-i}(x) + (-1)^q \ln(1-x) (Li_{p+q}(x) - \zeta(p+q)) - (-1)^q \sum_{n=1}^{\infty} \frac{1}{n^{p+q}} \left(\sum_{k=1}^n \frac{x^k}{k} \right). \quad (2.4)$$

Proof. Let

$$I_{p,q}(x) = \int_0^x \frac{Li_p(t) Li_q(t)}{t} dt. \quad (2.5)$$

By the definition of polylogarithm function, we can verify that

$$\int_0^x \frac{Li_p(t) Li_q(t)}{t} dt = \sum_{n=1}^{\infty} \frac{1}{n^p} \int_0^x t^{n-1} Li_q(t) dt. \quad (2.6)$$

is hold. Using integration by parts we have

$$\int_0^x t^{n-1} Li_q(t) dt = \sum_{i=1}^{q-1} (-1)^{i-1} \frac{x^n}{n^i} Li_{q+1-i}(x) + \frac{(-1)^q}{n^q} \ln(1-x) (x^n - 1) - \frac{(-1)^q}{n^q} \left(\sum_{k=1}^n \frac{x^k}{k} \right). \quad (2.7)$$

Combining (2.6) and (2.7), we obtain (2.4) holds.

Euler gave the following formula in 1775 (see [2, 12])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^k} = \frac{1}{2} \left\{ (k+2) \zeta(k+1) - \sum_{i=1}^{k-2} \zeta(k-i) \zeta(i+1) \right\}. \quad (2.8)$$

Letting $x \rightarrow 1$ in Theorem 2.1, we can get the following results

Corollary 2.3 ([13]) *Let p, q be integers with $p, q \geq 1$. Then the integrals $I_{p,q}(1)$ reduce to polynomials in zeta values:*

$$\begin{aligned} \int_0^1 \frac{Li_p(x) Li_q(x)}{x} dx &= \sum_{i=1}^{q-1} (-1)^{i-1} \zeta(q+1-i) \zeta(p+i) + (-1)^{q-1} \left(1 + \frac{p+q}{2} \right) \zeta(p+q+1) \\ &\quad - \frac{1}{2} \sum_{k=1}^{p+q-2} \zeta(k+1) \zeta(p+q-k). \end{aligned}$$

Theorem 2.4 *Let s, t be integers with $s, t \geq 1$, we have*

$$\begin{aligned} \int_0^1 \frac{Li_s^2(x) Li_t(x)}{x} dx &= 2 \sum_{j=1}^s A_j^{(s,s)} \left\{ \sum_{i=1}^{t-1} (-1)^{i-1} \zeta(t+1-i) \sum_{n=1}^{\infty} \frac{\zeta_n(j)}{n^{2s+i-j}} + (-1)^{t-1} \sum_{n=1}^{\infty} \frac{H_n \zeta_n(j)}{n^{2s+t-j}} \right\} \\ &\quad - 2 \sum_{j=1}^s A_j^{(s,s)} I(2s, t), \end{aligned} \quad (2.9)$$

$$\begin{aligned} \int_0^1 \frac{Li_s^2(x) Li_t(x)}{x} dx &= (-1)^{s-1} \sum_{n=1}^{\infty} \left\{ \sum_{j=1}^s A_j^{(s,t)} \frac{H_n \zeta_n(j)}{n^{2s+t-j}} + \sum_{j=1}^t B_j^{(s,t)} \frac{H_n \zeta_n(j)}{n^{2s+t-j}} \right\} \\ &\quad + \sum_{i=1}^{s-1} (-1)^{i-1} \zeta(s+1-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=1}^s A_j^{(s,t)} \frac{\zeta_n(j)}{n^{s+t+i-j}} + \sum_{j=1}^t B_j^{(s,t)} \frac{\zeta_n(j)}{n^{s+t+i-j}} \right\} \\ &\quad - \left\{ \sum_{j=1}^s A_j^{(s,t)} + \sum_{j=1}^t B_j^{(s,t)} \right\} I(s, s+t). \end{aligned} \quad (2.10)$$

Proof. Let $s = t$ in (2.1), we have

$$Li_s^2(x) = 2 \sum_{j=1}^s A_j^{(s,s)} \sum_{n=1}^{\infty} \frac{\zeta_n(j)}{n^{2s-j}} x^n - 2 \sum_{j=1}^s A_j^{(s,s)} Li_{2s}(x). \quad (2.11)$$

Multiplying $\frac{Li_t(x)}{x}$ to the equation (2.11) and integrating over $(0, 1)$, by virtue of (2.7), we obtain (2.9).

Similarly, multiplying $\frac{Li_s(x)}{x}$ in both sides of (2.1), and integrating from $x = 0$ to $x = 1$ with (2.7). By a simple calculation, we obtain formula (2.10).

Theorem 2.5 For $p \geq 2, m \geq 0$ and $p, m \in \mathbb{Z}, x \in [-1, 1)$, we have

$$\begin{aligned}
& (-1)^p \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(p+2m+1)}{n^p} + \frac{\zeta_n(p)}{n^{p+2m+1}} \right\} \left(\sum_{k=1}^n \frac{x^k}{k} \right) \\
&= (p+2m+2) I_{p-1, p+2m+2}(x) + (2m+1) I_{p, p+2m+1}(x) - (p+1) I_{p+1, p+2m}(x) \\
&+ (-1)^p \ln(1-x) \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(p+2m+1)}{n^p} + \frac{\zeta_n(p)}{n^{p+2m+1}} \right\} (x^n - 1) \\
&+ \sum_{i=1}^{p+2m} (-1)^{i-1} Li_{p+2m+2-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p-1} \frac{\zeta_n(j)}{n^{p+i-j}} + 2 \frac{H_n}{n^{p+i-1}} \right\} x^n \\
&+ \sum_{i=1}^{p+2m-1} (-1)^{i-1} Li_{p+2m+1-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{p+1+i-j}} + 2 \frac{H_n}{n^{p+i}} \right\} x^n \\
&- \sum_{i=1}^{p-2} (-1)^{i-1} Li_{p-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m+1} \frac{\zeta_n(j)}{n^{p+2m+2+i-j}} + 2 \frac{H_n}{n^{p+2m+1+i}} \right\} x^n \\
&- \sum_{i=1}^{p-1} (-1)^{i-1} Li_{p+1-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m} \frac{\zeta_n(j)}{n^{p+2m+1+i-j}} + 2 \frac{H_n}{n^{p+2m+i}} \right\} x^n. \tag{2.12}
\end{aligned}$$

where the integral $I_{p,q}(x)$ is defined by (2.5).

Proof. We first consider the following integral $\int_0^x \frac{\ln(1-t) Li_p(t) Li_{p+2m}(t)}{t} dt$, $x \in [-1, 1)$. In (2.1), taking $s=1, t=p$, we have that

$$\ln(1-t) Li_p(t) = (p+1) Li_{p+1}(t) - \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{p+1-j}} + 2 \frac{H_n}{n^p} \right\} t^n, \quad x \in [-1, 1). \tag{2.13}$$

By virtue of (2.7) and (2.13), we obtain

$$\begin{aligned}
& \int_0^x \frac{\ln(1-t) Li_p(t) Li_{p+2m}(t)}{t} dt \\
&= (p+1) I_{p+1, p+2m}(x) - \sum_{i=1}^{p+2m-1} (-1)^{i-1} Li_{p+2m+1-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{p+1+i-j}} + 2 \frac{H_n}{n^{p+i}} \right\} x^n \\
&+ (-1)^{p-1} \ln(1-x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{2p+2m+1-j}} + 2 \frac{H_n}{n^{2p+2m}} \right\} (x^n - 1) \\
&- (-1)^{p-1} \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{2p+2m+1-j}} + 2 \frac{H_n}{n^{2p+2m}} \right\} \left(\sum_{k=1}^n \frac{x^k}{k} \right) \\
&= (p+2m+1) I_{p, p+2m+1}(x) - \sum_{i=1}^{p-1} (-1)^{i-1} Li_{p+1-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m} \frac{\zeta_n(j)}{n^{p+2m+1+i-j}} + 2 \frac{H_n}{n^{p+2m+i}} \right\} x^n
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{p-1} \ln(1-x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m} \frac{\zeta_n(j)}{n^{2p+2m+1-j}} + 2 \frac{H_n}{n^{2p+2m}} \right\} (x^n - 1) \\
& - (-1)^{p-1} \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m} \frac{\zeta_n(j)}{n^{2p+2m+1-j}} + 2 \frac{H_n}{n^{2p+2m}} \right\} \left(\sum_{k=1}^n \frac{x^k}{k} \right). \tag{2.14}
\end{aligned}$$

After simplification, we find that

$$\begin{aligned}
& (-1)^{p-1} \sum_{n=1}^{\infty} \left\{ \sum_{j=p+1}^{p+2m} \frac{\zeta_n(j)}{n^{2p+2m+1-j}} \right\} \left(\sum_{k=1}^n \frac{x^k}{k} \right) \\
& = (p+2m+1) I_{p,p+2m+1}(x) - (p+1) I_{p+1,p+2m}(x) \\
& + (-1)^{p-1} \ln(1-x) \sum_{n=1}^{\infty} \left\{ \sum_{j=p+1}^{p+2m} \frac{\zeta_n(j)}{n^{2p+2m+1-j}} \right\} (x^n - 1) \\
& + \sum_{i=1}^{p+2m-1} (-1)^{i-1} Li_{p+2m+1-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{p+1+i-j}} + 2 \frac{H_n}{n^{p+i}} \right\} x^n \\
& - \sum_{i=1}^{p-1} (-1)^{i-1} Li_{p+1-i}(x) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m} \frac{\zeta_n(j)}{n^{p+2m+1+i-j}} + 2 \frac{H_n}{n^{p+2m+i}} \right\} x^n. \tag{2.15}
\end{aligned}$$

Replacing p by $p-1$, m by $m+1$ in (2.15), then combining with (2.15), we can obtain (2.12). Let $x = -1$ and $x \rightarrow 1$ in (2.12), we can gives the following Corollaries:

Corollary 2.6 For $p \geq 2, m \geq 0$ and $p, m \in \mathbb{Z}$, we have

$$\begin{aligned}
& (-1)^p \sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(p+2m+1)}{n^p} + \frac{H_n \zeta_n(p)}{n^{p+2m+1}} \right\} \\
& = (p+2m+2) I_{p-1,p+2m+2}(1) + (2m+1) I_{p,p+2m+1}(1) - (p+1) I_{p+1,p+2m}(1) \\
& + \sum_{i=1}^{p+2m} (-1)^{i-1} \zeta(p+2m+2-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p-1} \frac{\zeta_n(j)}{n^{p+i-j}} + 2 \frac{H_n}{n^{p-1+i}} \right\} \\
& + \sum_{i=1}^{p+2m-1} (-1)^{i-1} \zeta(p+2m+1-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{p+i+1-j}} + 2 \frac{H_n}{n^{p+i}} \right\} \\
& - \sum_{i=1}^{p-2} (-1)^{i-1} \zeta(p-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m+1} \frac{\zeta_n(j)}{n^{p+2m+i+2-j}} + 2 \frac{H_n}{n^{p+2m+1+i}} \right\} \\
& - \sum_{i=1}^{p-1} (-1)^{i-1} \zeta(p+1-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m} \frac{\zeta_n(j)}{n^{p+2m+i+1-j}} + 2 \frac{H_n}{n^{p+2m+i}} \right\}. \tag{2.16}
\end{aligned}$$

Corollary 2.7 For $p \geq 2, m \geq 0$ and $p, m \in \mathbb{Z}$, we have

$$(-1)^p \sum_{n=1}^{\infty} \left\{ \frac{L_n(1) \zeta_n(p+2m+1)}{n^p} + \frac{L_n(1) \zeta_n(p)}{n^{p+2m+1}} \right\}$$

$$\begin{aligned}
&= (p+1) I_{p+1,p+2m}(-1) - (2m+1) I_{p,p+2m+1}(-1) - (p+2m+2) I_{p-1,p+2m+2}(-1) \\
&+ (-1)^p \ln 2 \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(p+2m+1)}{n^p} + \frac{\zeta_n(p)}{n^{p+2m+1}} \right\} \left((-1)^{n-1} + 1 \right) \\
&+ \sum_{i=1}^{p-1} (-1)^{i-1} \bar{\zeta}(p+1-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m} \frac{\zeta_n(j)}{n^{p+2m+1+i-j}} + 2 \frac{H_n}{n^{p+2m+i}} \right\} (-1)^{n-1} \\
&+ \sum_{i=1}^{p-2} (-1)^{i-1} \bar{\zeta}(p-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m+1} \frac{\zeta_n(j)}{n^{p+2m+2+i-j}} + 2 \frac{H_n}{n^{p+2m+1+i}} \right\} (-1)^{n-1} \\
&- \sum_{i=1}^{p+2m-1} (-1)^{i-1} \bar{\zeta}(p+2m+1-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{p+1+i-j}} + 2 \frac{H_n}{n^{p+i}} \right\} (-1)^{n-1} \\
&- \sum_{i=1}^{p+2m} (-1)^{i-1} \bar{\zeta}(p+2m+2-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p-1} \frac{\zeta_n(j)}{n^{p+i-j}} + 2 \frac{H_n}{n^{p+i-1}} \right\} (-1)^{n-1}. \tag{2.17}
\end{aligned}$$

In the same manner, we obtain the following Theorem:

Theorem 2.8 For $p \geq 2, m \geq 0$ and $p, m \in \mathbb{Z}$, we have

$$\begin{aligned}
&(-1)^p \sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(p+2m+2)}{n^p} - \frac{H_n \zeta_n(p)}{n^{p+2m+2}} \right\} \\
&= \sum_{i=1}^{p+2m+1} (-1)^{i-1} \zeta(p+2m+3-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p-1} \frac{\zeta_n(j)}{n^{p+i-j}} + 2 \frac{H_n}{n^{p-1+i}} \right\} \\
&+ \sum_{i=1}^{p+2m} (-1)^{i-1} \zeta(p+2m+2-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^p \frac{\zeta_n(j)}{n^{p+i+1-j}} + 2 \frac{H_n}{n^{p+i}} \right\} \\
&- \sum_{i=1}^{p-2} (-1)^{i-1} \zeta(p-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m+2} \frac{\zeta_n(j)}{n^{p+2m+i+3-j}} + 2 \frac{H_n}{n^{p+2m+2+i}} \right\} \\
&- \sum_{i=1}^{p-1} (-1)^{i-1} \zeta(p+1-i) \sum_{n=1}^{\infty} \left\{ \sum_{j=2}^{p+2m+1} \frac{\zeta_n(j)}{n^{p+2m+i+2-j}} + 2 \frac{H_n}{n^{p+2m+i+1}} \right\} \\
&+ (p+2m+3) I(p-1, p+2m+3) + (2m+2) I(p, p+2m+2) \\
&- (p+1) I(p+1, p+2m+1). \tag{2.18}
\end{aligned}$$

Proof. Similarly to the proof of Theorem 2.5, considering integral $\int_0^1 \frac{\ln(1-x) Li_p(x) Li_{p+2m+1}(x)}{x} dx$, we deduce Theorem 2.8 holds.

Theorem 2.9 For $p \geq 2, m \geq 0$ and $p, m \in \mathbb{Z}$, we have

$$(-1)^p \sum_{n=1}^{\infty} \left\{ \frac{L_n(1) L_n(p+2m+1)}{n^p} + \frac{L_n(1) L_n(p)}{n^{p+2m+1}} \right\} (-1)^n$$

$$\begin{aligned}
&= (p+2m+1) I_{p+2m+2,p-1}(-1) + R_{p+2m+2,p-1}(-1) - (p-1) I_{p,p+2m+1}(-1) - R_{p,p+2m+1}(-1) \\
&\quad + (p+2m) I_{p+2m+1,p}(-1) + R_{p+2m+1,p}(-1) - p I_{p+1,p+2m}(-1) - R_{p+1,p+2m}(-1) \\
&\quad + (-1)^p \ln 2 \sum_{n=1}^{\infty} \left\{ \frac{L_n(p+2m+1)}{n^p} + \frac{L_n(p)}{n^{p+2m+1}} \right\} ((-1)^n - 1) \\
&\quad + \sum_{i=1}^{p-1} (-1)^{i-1} \bar{\zeta}(p+1-i) \sum_{n=1}^{\infty} \left\{ \frac{L_n(1)}{n^{p+2m+i}} (-1)^{n-1} - \sum_{j=2}^{p+2m} \frac{L_n(j)}{n^{p+2m+1+i-j}} \right\} \\
&\quad + \sum_{i=1}^{p-2} (-1)^{i-1} \bar{\zeta}(p-i) \sum_{n=1}^{\infty} \left\{ \frac{L_n(1)}{n^{p+2m+1+i}} (-1)^{n-1} - \sum_{j=2}^{p+2m+1} \frac{L_n(j)}{n^{p+2m+2+i-j}} \right\} \\
&\quad - \sum_{i=1}^{p+2m} (-1)^{i-1} \bar{\zeta}(p+2m+2-i) \sum_{n=1}^{\infty} \left\{ \frac{L_n(1)}{n^{p+i-1}} (-1)^{n-1} - \sum_{j=2}^{p-1} \frac{L_n(j)}{n^{p+i-j}} \right\} \\
&\quad - \sum_{i=1}^{p+2m-1} (-1)^{i-1} \bar{\zeta}(p+2m+1-i) \sum_{n=1}^{\infty} \left\{ \frac{L_n(1)}{n^{p+i}} (-1)^{n-1} - \sum_{j=2}^p \frac{L_n(j)}{n^{p+1+i-j}} \right\}. \tag{2.19}
\end{aligned}$$

where $R_{p,q}(x) = \int_0^x \frac{Li_p(-t) Li_q(t)}{t} dt$ and

$$\begin{aligned}
R_{p,q}(x) &= \sum_{i=1}^{q-1} (-1)^{i-1} Li_{p+i}(-x) Li_{q+1-i}(x) + (-1)^q \ln(1-x) (Li_{p+q}(-x) + \bar{\zeta}(p+q)) \\
&\quad - (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{p+q}} \left(\sum_{k=1}^n \frac{x^k}{k} \right). \tag{2.20}
\end{aligned}$$

For example

$$\begin{aligned}
R_{4,1}(-1) &= \sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} (-1)^{n-1} - \frac{31}{16} \zeta(5) \ln 2, \\
R_{2,3}(-1) &= \sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} (-1)^{n-1} + \frac{7}{8} \zeta(6) - \frac{3}{4} \zeta^2(3) - \frac{31}{16} \zeta(5) \ln 2.
\end{aligned}$$

Proof. Similarly to the proof of Theorem 2.5 and 2.8, we consider the following integral

$$\int_0^{-1} \frac{\ln(1+x) Li_p(x) Li_{p+2m}(x)}{x} dx.$$

Noting that

$$\ln(1+x) Li_p(x) = p Li_{p+1}(x) + Li_{p+1}(-x) + \sum_{n=1}^{\infty} \frac{L_n(1)}{n^p} x^n + \sum_{i=1}^p \sum_{n=1}^{\infty} \frac{L_n(i)}{n^{p+1-i}} (-x)^n. \tag{2.21}$$

After simplification, we deduce Theorem 2.9 holds.

3 Representation of Euler sums by zeta values and linear sums

In this section, we consider the analytic representations of two quadratic Euler sums which involves harmonic numbers and alternating harmonic numbers

$$\sum_{n=1}^{\infty} \frac{\zeta_n(p) \zeta_n(p+2m+1)}{n} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{L_n(p) L_n(p+2m+1)}{n} (-1)^{n-1}$$

through zeta values and linear sums, and give explicit formulae for several 6-order quadratic sums in terms of zeta values and linear sums.

Theorem 3.1 *For $1 \leq l_1, l_2, m \in \mathbb{Z}$ and $x, y, z \in [-1, 1)$, we have the following relation*

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\zeta_n(l_1, x) \zeta_n(l_2, y)}{n^m} z^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1, x) \zeta_n(m, z)}{n^{l_2}} y^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2, y) \zeta_n(m, z)}{n^{l_1}} x^n \\ &= \sum_{n=1}^{\infty} \frac{\zeta_n(m, z)}{n^{l_1+l_2}} (xy)^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1, x)}{n^{m+l_2}} (yz)^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2, y)}{n^{l_1+m}} (xz)^n \\ &+ Li_m(z) Li_{l_1}(x) Li_{l_2}(y) - Li_{l_1+l_2+m}(xyz) \end{aligned} \quad (3.1)$$

where the partial sum $\zeta_n(l, x)$ is defined by $\zeta_n(l, x) = \sum_{k=1}^n \frac{x^k}{k^l}$.

Proof. First, we construct the function $F(x, y, z) = \sum_{n=1}^{\infty} \{\zeta_n(l_1, x) \zeta_n(l_2, y) - \zeta_n(l_1 + l_2, xy)\} z^{n-1}$.

By the definition of $\zeta_n(l, x)$, we have

$$F(x, y, z) = zF(x, y, z) + \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1, x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2, y)}{(n+1)^{l_1}} x^{n+1} \right\} z^n. \quad (3.2)$$

Moving $zF(x, y, z)$ from right to left and then multiplying $(1-z)^{-1}$ to the equation (3.2) and integrating over the interval $(0, z)$, we obtain

$$\sum_{n=1}^{\infty} \frac{\zeta_n(l_1, x) \zeta_n(l_2, y) - \zeta_n(l_1 + l_2, xy)}{n} z^n = \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1, x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2, y)}{(n+1)^{l_1}} x^{n+1} \right\} \{Li_{l_1}(z) - \zeta_n(1, z)\} \quad (3.3)$$

Furthermore, using integration and the following formula

$$\sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1, x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2, y)}{(n+1)^{l_1}} x^{n+1} \right\} = Li_{l_1}(x) Li_{l_2}(y) - Li_{l_1+l_2}(xy)$$

we can obtain (3.1).

Let $(x, y, z) = (-1, 1, 1)$, $(l_1, l_2, m) = (1, p+2m+1, p)$ and $(x, y, z) = (-1, -1, -1)$, $(l_1, l_2, m) = (1, p+2m+1, p)$ in (3.1), we can give the following Corollaries

Corollary 3.2 *If $1 < p \in \mathbb{Z}$, $0 \leq m \in \mathbb{Z}$, then we have*

$$\sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(p+2m+1)}{n^p} + \sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(p)}{n^{p+2m+1}} + \sum_{n=1}^{\infty} \frac{\zeta_n(p+2m+1) \zeta_n(p)}{n} (-1)^{n-1}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^{p+2m+2}} (-1)^{n-1} + \sum_{n=1}^{\infty} \frac{L_n(1)}{n^{2p+2m+1}} + \sum_{n=1}^{\infty} \frac{\zeta_n(p+2m+1)}{n^{p+1}} (-1)^{n-1} \\
&\quad + \ln 2 \zeta(p+2m+1) \zeta(p) - \bar{\zeta}(2p+2m+2).
\end{aligned} \tag{3.4}$$

Corollary 3.3 *If $1 \leq p \in Z, 0 \leq m \in Z$, then we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{L_n(1) L_n(p+2m+1)}{n^p} (-1)^{n-1} + \sum_{n=1}^{\infty} \frac{L_n(1) L_n(p)}{n^{p+2m+1}} (-1)^{n-1} + \sum_{n=1}^{\infty} \frac{L_n(p+2m+1) L_n(p)}{n} (-1)^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{L_n(p)}{n^{p+2m+2}} + \sum_{n=1}^{\infty} \frac{L_n(1)}{n^{2p+2m+1}} + \sum_{n=1}^{\infty} \frac{L_n(p+2m+1)}{n^{p+1}} \\
&\quad + \ln 2 \bar{\zeta}(p+2m+1) \bar{\zeta}(p) - \bar{\zeta}(2p+2m+2).
\end{aligned} \tag{3.5}$$

From Corollary 2.7, 3.2, 3.3 and Theorem 2.9, we can obtain the following Theorem

Theorem 3.4 *For $2 \leq p \in Z$ and $0 \leq m \in Z$, the quadratic sums*

$$\sum_{n=1}^{\infty} \frac{\zeta_n(p) \zeta_n(p+2m+1)}{n} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{L_n(p) L_n(p+2m+1)}{n} (-1)^{n-1}$$

are reducible to linear sums.

Using mathematica, we can obtain the following numerical values

$$\begin{aligned}
Li_4\left(\frac{1}{2}\right) &= 0.5174790616738993863307581618988629456, \\
\sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} (-1)^{n-1} &= 0.987441426403299713771650007985, \\
\sum_{n=1}^{\infty} \frac{L_n(2)}{n^4} &= 1.06358224101814909880154833539, \\
\sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^4} (-1)^{n-1} &= 0.934707899349253255197542851216, \\
\sum_{n=1}^{\infty} \frac{H_n}{n^5} (-1)^{n-1} &= 0.959151942504318157165421137321, \\
\sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} &= 1.02005194570145237930331996837, \\
\sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(2)}{n^3} &= 1.15935334356951415975457027807, \\
\sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(3)}{n^2} &= 1.47723102170162037670053143416,
\end{aligned}$$

Taking $p = 2, m = 0$ in (2.17)(2.19)(3.4)(3.5), we obtain

$$\sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(3)}{n^2} + \sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(2)}{n^3}$$

$$\begin{aligned}
&= \frac{3}{4}\zeta^2(3) + \frac{7}{4}\zeta(6) + \frac{5}{8}\zeta(2)\zeta(3)\ln 2 - 2\zeta(2) Li_4\left(\frac{1}{2}\right) + \frac{5}{4}\zeta(4)\ln^2 2 \\
&\quad - \frac{1}{12}\zeta(2)\ln^4 2,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\zeta_n(2)\zeta_n(3)}{n} (-1)^{n-1} \\
&= -\frac{161}{64}\zeta(6) + \frac{31}{16}\zeta(5)\ln 2 + \frac{9}{32}\zeta^2(3) + \frac{3}{8}\zeta(2)\zeta(3)\ln 2 + 2\zeta(2) Li_4\left(\frac{1}{2}\right) \\
&\quad - \frac{5}{4}\zeta(4)\ln^2 2 + \frac{1}{12}\zeta(2)\ln^4 2 + \sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^4} (-1)^{n-1} - \sum_{n=1}^{\infty} \frac{L_n(3)}{n^3},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{L_n(1)L_n(3)}{n^2} (-1)^{n-1} + \sum_{n=1}^{\infty} \frac{L_n(1)L_n(2)}{n^3} (-1)^{n-1} \\
&= -\frac{385}{128}\zeta(6) + \frac{31}{8}\zeta(5)\ln 2 + \frac{3}{32}\zeta^2(3) + \frac{9}{8}\zeta(2)\zeta(3)\ln 2 + \zeta(2) Li_4\left(\frac{1}{2}\right) \\
&\quad - \frac{5}{8}\zeta(4)\ln^2 2 + \frac{1}{24}\zeta(2)\ln^4 2
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{L_n(2)L_n(3)}{n} (-1)^{n-1} \\
&= \frac{163}{128}\zeta(6) - \frac{31}{16}\zeta(5)\ln 2 + \frac{3}{16}\zeta^2(3) - \frac{3}{4}\zeta(2)\zeta(3)\ln 2 - \zeta(2) Li_4\left(\frac{1}{2}\right) \\
&\quad + \frac{5}{8}\zeta(4)\ln^2 2 - \frac{1}{24}\zeta(2)\ln^4 2 + \sum_{n=1}^{\infty} \frac{L_n(2)}{n^4} + \sum_{n=1}^{\infty} \frac{L_n(3)}{n^3}.
\end{aligned} \tag{3.9}$$

In [21], we gave the following formula

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L_n(1)\zeta_n(2)}{n^3} &= \frac{29}{8}\zeta(2)\zeta(3)\ln 2 - \frac{93}{32}\zeta(5)\ln 2 - \frac{1855}{128}\zeta(6) + \frac{17}{16}\zeta^2(3) - \sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} (-1)^{n-1} \\
&\quad + \sum_{n=1}^{\infty} \frac{L_n(2)}{n^4} + 4 \sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^4} (-1)^{n-1} + 8 \sum_{n=1}^{\infty} \frac{H_n}{n^5} (-1)^{n-1}.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.6) respectively, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L_n(1)\zeta_n(3)}{n^2} &= \frac{2079}{128}\zeta(6) + \frac{93}{32}\zeta(5)\ln 2 - \frac{5}{16}\zeta^2(3) - 3\zeta(2)\zeta(3)\ln 2 - 2\zeta(2) Li_4\left(\frac{1}{2}\right) \\
&\quad + \frac{5}{4}\zeta(4)\ln^2 2 - \frac{1}{12}\zeta(2)\ln^4 2 + \sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} (-1)^{n-1} - \sum_{n=1}^{\infty} \frac{L_n(2)}{n^4} \\
&\quad - 4 \sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^4} (-1)^{n-1} - 8 \sum_{n=1}^{\infty} \frac{H_n}{n^5} (-1)^{n-1}.
\end{aligned} \tag{3.11}$$

Proceeding in a similar fashion to evaluation of the Theorem 2.5, it is possible to evaluate other Euler sums involving harmonic numbers and alternating harmonic numbers. For example

$$\sum_{n=1}^{\infty} \left\{ \frac{H_n L_n(1)}{n^3} - \frac{H_n L_n(3)}{n} \right\} (-1)^{n-1} = \frac{15}{4} \zeta(4) \ln 2 - \frac{9}{8} \zeta(2) \zeta(3) - \frac{1}{2} \zeta(3) \ln^2 2.$$

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References

- [1] George E. Andrews, Richard Askey, Ranjan Roy. *Special Functions*. Cambridge University Press., 2000: 481-532.
- [2] David H. Bailey, Jonathan M. Borwein and Roland Girgensohn. *Experimental Evaluation of Euler Sums*. Experimental Mathematics., 1994, **3**(1): 17-30.
- [3] David H. Bailey, Jonathan M. Borwein, Richard E. Crandall. *Computation and theory of extended Mordell-Tornheim-Witten sums*. Math. Comp., 2014, **83**(288): 1795-1821.
- [4] B. C. Berndt. *Ramanujans Notebooks, Part I*. Springer-Verlag, New York., 1985.
- [5] B. C. Berndt. *Ramanujans Notebooks, Part II*. Springer-Verlag, New York., 1989.
- [6] David Borwein, Jonathan M. Borwein and Roland Girgensohn. *Explicit Evaluation of Euler Sums*. Proc. Edinburgh Math., 1995, **38**: 277-294.
- [7] J.Borwein, P.Borwein, R.Girgensohn, S.Parnes. *Making Sense of Experimental Mathematics*. Mathematical Intelligencer., 1996, **18**(4): 12-18.
- [8] Jonathan M. Borwein, David M. Bradley, David J. Broadhurst, Petr. Lisonk. *Special values of multiple polylogarithms*. Trans. Amer. Math. Soc., 2001, **353**(3): 907-941.
- [9] J. M. Borwein, I. J. Zucker, J. Boersma. *The evaluation of character Euler double sums*. Ramanujan J., 2008, **15** (3): 377-405.
- [10] J.M. Borwein, R. Girgensohn, *Evaluation of Triple Euler Sums*, Electron. J. Combin., 1996: 2-7.
- [11] Minking Eie, Chuan-Sheng Wei. *Evaluations of Some Quadruple Euler Sums of even Weight*. Functions et Approximatio., 2012, **46**(1): 63-67.
- [12] Philippe Flajolet and Bruno Salvy. *Euler Sums and Contour Integral Representations*. Experimental Mathematics., 1998, **7**(1): 15-35.
- [13] Pedro freitas. *Integrals of Polylogarithmic Functions, Recurrence Relations, and Associated Euler Sums*. Mathematics of Computation., 2005, **74**(251): 1425-1440.
- [14] Comtet L. Advanced combinatorics, Boston: D Reidel Publishing Company, 1974.
- [15] I. Mezö. *Nonlinear Euler sums*. Pacific J. Math., 2014, **272**: 201-226.
- [16] A. Sofo. *Integral forms of sums associated with harmonic numbers*. Applied Mathematics and Computation., 2009, **207**(2): 365-372.
- [17] A. Sofo. *Quadratic alternating harmonic number sums*. J. Number Theory., 2015, **154**: 144-159.
- [18] C. Markett. *Triple Sums and the Riemann Zeta Function*. J. Number Theory. 1994, **48**(2): 113-132.
- [19] Ce Xu, Jinfa Cheng. *Some Results On Euler Sums*. Functions et Approximatio., 2016, **54**(1): 25-37.
- [20] Ce Xu, Yuhuan Yan, Zhijuan Shi. *Euler sums and integrals of polylogarithm functions*. J. Number Theory., 2016, **165**(6): 84-108.
- [21] Ce Xu, Yingyue Yang, Jianwen Zhang. *Explicit evaluation of quadratic Euler sums*. Int. J. Number Theory., 2016, in press.